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A Graph Theoretic Approach to the Investigation of System-Environment Relationships[1]

The purpose of this paper is to undertake a graph theoretic analysis of those empirical structures for which it is meaningful to consider a part of the structure as a system and the remainder as its environment. We shall be primarily concerned with social systems such as groups, social networks or organizations, but the results are applicable to any structure that can be appropriately represented by means of graph theory.

The term *empirical structure* is here used to refer to a set of empirical elements (such as people, roles or positions) together with a set of empirical relationships (such as liking, communication or influence) between pairs of elements. Given such a structure, its digraph D is obtained as follows. The set of points of D, denoted $V = \{v_1, v_2, \ldots, v_p\}$, corresponds to the set of empirical elements, and there is an arc (directed line) from v_i to v_j if and only if the corresponding ordered pair of elements is in the specified empirical relation. Throughout the following discussion we shall be concerned with relationships that can be viewed as "links" of communication by which "messages" are transmitted from one element to another. In graph theory, a *walk* W of D is defined as an alternating sequence of points and arcs which begins and ends with a point and has the property that each arc is preceded by its first point and followed by its second one. Thus each walk of D represents a "chain of communication" within the structure, and it is possible for a message to reach element v_j from v_i if and only if D contains a walk from v_i to v_j.

It should be noted that the term *message* is used here to refer to anything that can be transmitted from one element to another. Thus, for example, if the elements of a structure are thought of as roles, subgroups or positions of an organization, then a message might be a person, memorandum, unit of work or some other "object" that can change its location in the organization. If, on the other hand, elements are taken to be individual people, then a message might be any of the following: (a) an item of information, an opinion or a rumor; (b) an influence attempt, such as an order, request or suggestion; (c) some ma-

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terial object that may be given, lent or sold; or (d) some symbolic object such as a favor, approval, help or support. This latter type of interpretation has been employed in research on social networks, as reported by Mitchell (1969) and Barnes (1972), where it is assumed that a variety of such messages can be transmitted through a given network. A link that carries more than one kind of message is said by these authors to be "multistranded."

Let us assume that we have some empirical basis for identifying a particular system within a given structure so that the set V of points of its associated digraph can be partitioned into two subsets, $S = \{s_1, s_2, \ldots, s_m\}$, corresponding to the elements of the system, and $E = \{e_1, e_2, \ldots, e_{p-m}\} = V - S$, corresponding to the elements of its environment. Strictly speaking, a *system* and its *environment* correspond to the subgraphs $\langle S \rangle$ and $\langle E \rangle$ that are induced by their sets of points,[2] but it will be convenient to denote them more simply by S and E.

For illustrative purposes, two rather different sorts of interpretation will be employed throughout this paper. In the first, we assume that a digraph D represents the *interpersonal communication structure* of a group of people and that system S is a specific subgroup within the group. Messages are taken to be items of information which can be transmitted only via links of the structure. Thus, points correspond to individual people, arcs correspond to links of interpersonal communication and walks indicate permissible chains of communication. In the second interpretation, we assume that D represents the *career structure* of an organization and that S represents a specified department within the organization. Here, points correspond to positions (or jobs), messages are thought of as individual people, arcs indicate permissible changes of position and walks designate permissible career lines of individuals.

Figure 1 shows the digraph of a small structure containing a system, $S = \{s_1, s_2, s_3\}$, and its environment, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. It can readily be seen that in this digraph every walk W between a point of S and a point of E must contain arcs s_1e_1 or e_1s_1 and hence points s_1 and e_1. The corresponding communication links and elements of the structure are thus uniquely involved in all transactions between the system and its environment and would seem intuitively to be located in the "boundary" between S and E.

If this digraph is interpreted as an interpersonal communication structure, then all chains of communication between a member of subgroup S and a nonmember must contain "boundary" persons s_1 and e_1 who are uniquely able to monitor, modify or intercept any item of information going between the subgroup and its environment. It is also reasonable to assume, in keeping with the analysis of the functions of gossip in social networks presented by Epstein (1969), that when an item of information traverses a "boundary link" of such

[2]The definitions of concepts of graph theory that are not given in this paper may be found in Harary, Norman and Cartwright (1965).

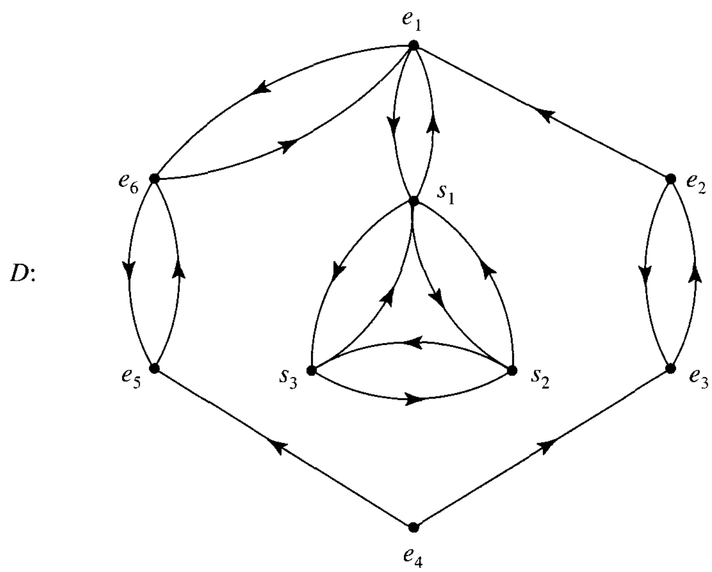


Figure 1. The digraph of a system $S = \{s_1, s_2, s_3\}$ and its environment $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

a system, its content or meaning may be significantly altered. In a similar way, if D is interpreted as the career structure of an organization, then all career lines that include positions in both department S and its environment must contain "boundary positions" s_1 and e_1. Anyone entering or leaving the department can do so only by going directly from one of these positions to the other, and when a person traverses a "boundary link" in the career structure, he changes departmental membership and consequently becomes subjected to a different set of influences and expectations.

The concept of boundary has been employed extensively by theorists (Emery, 1969; Katz and Kahn, 1966; Miller and Rice, 1967; Rice, 1963) who adopt a "systems approach" to social organizations. According to this view, those members of an organization who are directly involved in transactions between the organization and its environment are said to occupy "boundary roles" and various consequences have been identified for such individuals. Thus, for example, Kahn et al. (1964) have reported that occupants of boundary roles are more likely to experience role conflict than are members located "deep within the organization." The explanation proposed by Kahn et al. to account for this difference can be elucidated by means of the digraph shown in Figure 1. Let us assume that S and E represent a (miniature) organization and its environment, that the points of D correspond to individuals, that role expectations are communicated via the arcs of D and that these expectations differ when they come from inside as opposed to outside the organization. It then follows immediately that person s_1 should experience more role conflict than s_2 or s_3.

The concept of boundary has also been used implicitly by Korte and Milgram (1970) in a study of acquaintance networks connecting racial groups in the United States. The procedure involved asking a number of "source" individuals, who were white, to try to send a booklet via a sequence of acquaintances to a black "target" person in a different community. The investigators were able to record the progress of each booklet as it proceeded through the network and thus to identify a chain of communication from each source to the target. If we conceive of the white segment of an acquaintance network as system S and the black segment as environment E, then every successful chain must contain a "boundary link" of S. One striking feature of this study is that such links tended to appear only late in each chain of communication; in 23 of the 35 completed chains, the only boundary link was the last one of the chain, and in seven it was next to the last. Korte and Milgram call the white member of such a link a "gatekeeper," a term introduced by Lewin (1943: 186) to refer more generally to anyone who can control the flow of objects across the boundary of a system.

In the following pages, we first develop a number of concepts which give precise meaning to the intuitive notions of the "boundary" of a system and the "degree of stratification" of a system and its environment. We then consider the properties of boundaries of systems contained in three common types of digraphs and conclude by showing how the concept of convexity can be used in the analysis of system-environment relationships.

Properties of Boundaries

From the foregoing discussion it is evident that the processes at the boundary between a system S and its environment E involve (a) *links* between the elements of S and E, (b) *elements of* S that are in direct contact with E and (c) *elements of* E that are in direct contact with S. The formal definition of boundary to be presented below makes use of the corresponding graph theoretical terms (arcs of D, points of S and points of E), but its statement requires some preliminary definitions which permit a distinction between links from S to E and those from E to S.

For any system S within a digraph D, its set of *out-liaisons*, denoted L_o, consists of those arcs v_iv_j of D from a point v_i of S to a point v_j of E, and its set L_i of *in-liaisons* consists of the arcs to a point of S from a point of E. Thus, every out-liaison is of the form $s_i e_j$ and every in-liaison is of the form $e_j s_i$. The out-liaisons and in-liaisons, taken together, constitute the set L of *liaisons* of S. The *out-frontier* F_o of S is the set of first points of all out-liaisons of S, the *in-frontier* F_i is the set of second points of all in-liaisons, and the *frontier* F is the union of F_o and F_i. The *out-neighborhood* N_o of S is the set of second points of all out-liaisons of S, the *in-neighborhood* N_i is the set of first points of all in-liaisons and the *neighborhood* N is the union of N_o and N_i.

With this background, we may now state a formal definition of the boundary of a system. The *boundary* $B(S, D)$ of a system S within a digraph D is the subgraph of D induced by the liaisons of S. In digraph theory, a *bipartite digraph* has its points partitioned into two subsets, V_1 and V_2, such that every arc joins a point of V_1 with a point of V_2. Thus, the boundary $B(S, D)$ is that bipartite subgraph of D in which the points of one set are all the points in the frontier of S, the points of the second set are all those in the neighborhood of S and the arcs are the liaisons of S.

In Figure 2, the boundary $B(S, D)$, for system $S = \{s_1, s_2, s_3, s_4\}$, is displayed below the digraph D. It can readily be seen that its two sets of liaisons L_o and L_i, its out-frontier F_o and its in-frontier F_i, and its out-neighborhood N_o and in-neighborhood N_i are $L_o = \{s_3e_1, s_4e_2\}$, $L_i = \{e_1s_3, e_3s_1\}$, $F_o = \{s_3, s_4\}$, $F_i = \{s_1, s_3\}$, $N_o = \{e_1, e_2\}$, $N_i = \{e_1, e_3\}$. It will be noted that in this particular case the point s_3 lies in both the out-frontier and in-frontier of S and that e_1 lies in both the out-neighborhood and in-neighborhood of S.

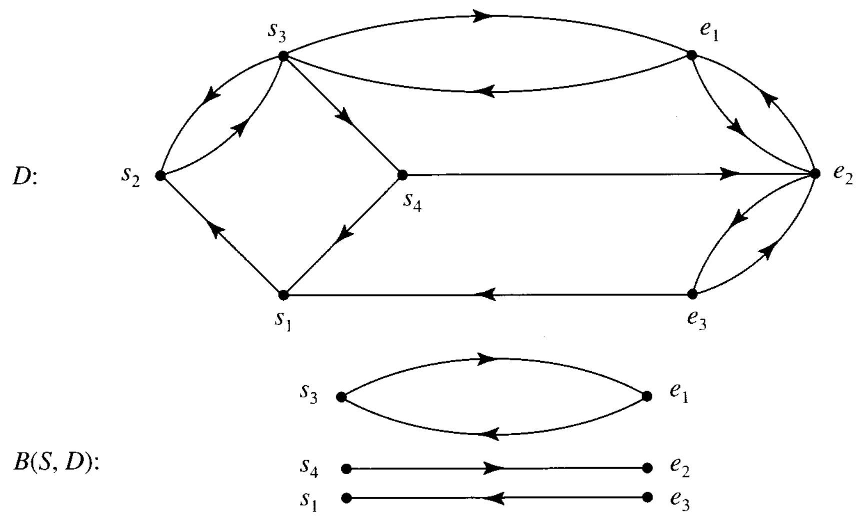


Figure 2. A system S and its boundary $B(S, D)$.

If the digraph of Figure 2 is interpreted as an interpersonal communication structure, then we see that the boundary of subgroup S is made up of (a) its members s_1, s_3 and s_4 who are located in the frontier of S, (b) nonmembers e_1, e_2 and e_3 who are in its neighborhood and (c) communication links s_3e_1, s_4e_2, e_1s_3 and e_3s_1 which join members and nonmembers. The out-frontier of the subgroup consists of all members who can communicate directly to at least one nonmember, and the in-frontier consists of those members who can receive a message directly from the outside. The out-neighborhood is composed of those people in the environment of the subgroup who can receive a message directly from a member, and its in-neighborhood is composed of those in the environment who can transmit one directly to a member. The out-liaisons consist of those communication links by which a message can leave S, whereas its in-liaisons are those by which a message can enter S.

If, on the other hand, the digraph is interpreted as the career structure of an organization, then $B(S, D)$ represents the boundary of department S. Its out-frontier consists of those positions from which a person may leave the department, and its in-frontier consists of those positions at which a person may enter. Its out-neighborhood consists of the positions outside the department to which a member may move, and its in-neighborhood consists of those from which a person may enter the department. Finally, the out-liaisons indicate permissible changes of position that take a person out of the department, and the in-liaisons indicate those by which a person may enter.

We shall refer to the sets F_o, F_i, F, N_o, N_i, N, L_o, L_i and L as the *boundary sets* of $B(S, D)$ and denote their cardinalities by f_o, f_i, f, n_o, n_i, n, l_o, l_i and l, respectively. The minimum cardinality of each of these sets is clearly zero, in which case the set is empty. If we denote the number of points in S and D by m and p, then for the frontier sets F_o, F_i and F their maximum cardinality is m, for the neighborhood sets N_o, N_i and N it is $p - m$, for the liaison sets L_o and L_i it is $m(p - m)$ and for L it is $2m(p - m)$.

The *absolute size* of a boundary set is defined as its cardinality. Loosely speaking, a system may be said to be in closer contact with its environment the larger the size of its frontier, neighborhood, or liaison set. The *relative size* of F_o, F_i and F is defined as f_o/m, f_i/m and f/m. These ratios vary between 0 and 1 and indicate the proportion of elements of S that are in its out-frontier, in-frontier and frontier, respectively. The corresponding indices of the *relative size* of N_o, N_i and N are $n_o/(p - m)$, $n_i/(p - m)$ and $n/(p - m)$ and have analogous meanings. The *relative size* of L_o, L_i and L is given by $l_o/m(p - m)$, $l_i/m(p - m)$ and $l/2m(p - m)$. All of these ratios equal zero when all of the boundary sets are empty and all equal one when S is in maximal contact with its environment. For the structure shown in Figure 2, we see that the relative sizes of F_o, F_i and F are $2/4$, $2/4$ and $3/4$, the relative sizes of N_o, N_i and N are $2/3$, $2/3$

and $^{3}/_{3}$, the relative sizes of L_o and L_i are each $^{2}/_{12}$ and the relative size of L is $^{4}/_{24}$.

If the cardinality of each boundary set of a system is maximum, then every point of S is in both its out-frontier and in-frontier, every point of E is in both the out-neighborhood and in-neighborhood of S and there is an out-liaison from every point of S to every point of E and an in-liaison to every point of S from every point of E. In other words, every element of S can engage in direct two-way communication with every element of E.

When all of the boundary sets of S are empty, we say that S has no boundary. Such a situation obviously arises when $S = V$, since S then has no environment within D. And it can be easily shown that S has no boundary if and only if S is a union of weak components of D. A system with no boundary is completely self-contained with respect to a given structure either because it has no environment within the structure or because it has no contact with its environment.

If none of the boundary sets of S is empty, then the frontier F of S is as small as possible when it contains just one point s. It follows at once that both F_o and F_i must consist of the single point s. Similarly, its neighborhood N is of minimum size when it contains just one point e, which is then also the only point in N_o and N_i. When both of these conditions are met, the boundary consists of one point s in S, the point e in E, and the symmetric pair of liaisons joining them. Such a structure is shown in Figure 1.

It can be readily seen that if L_o is empty, then so are F_o and N_o. In this case, no message originating in the system can reach the environment. An example of this sort is provided in the theory of Markov chains, where points correspond to "states" and there is an arc v_iv_j in D if and only if the probability that there will be a direct transition from state v_i to v_j is greater than zero. A subset S of the points of a Markov chain is said to be "closed" if S has no out-liaisons (see Feller, 1957:349). Thus, when a Markov chain is closed, there can be no transition from a state in S to one not in S. If such a system has any liaisons, its boundary consists of the nonempty sets L_i, F_i and N_i. All of its liaisons thus go from E to S and we say that the boundary of S has an *inward orientation.* Figure 3 shows a system whose boundary has such an orientation.

If the boundary of a system contains out-liaisons but no in-liaisons, we say that it has an *outward orientation*. Since, in this case, all liaisons go from S to E, it follows that S can influence its environment but not be influenced by it.

We conclude this section by considering two indices for characterizing each element s of a system S. The *out-degree* $od_D(v)$ of a point v in a digraph D is the number of arcs from v, and the *in-degree* $id_D(v)$ is the number of arcs to v. We employ the notation $od_s(s)$ and $id_s(s)$ to denote the out-degree and in-degree of a point s within the subgraph S. These terms give the number of internal arcs that terminate at s. Similarly, the notation $od_B(s)$ and $id_B(s)$ refers to the

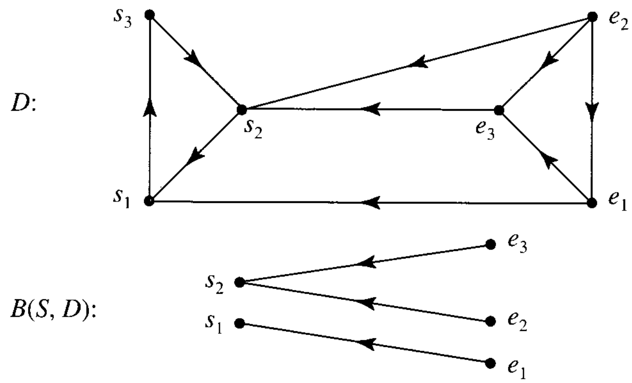


Figure 3. A boundary with an inward orientation.

boundary subgraph $B(S, D)$ and refers to the liaisons incident with s. Since every arc of D that is incident with s is either an internal arc or a liaison, it follows that:

$$od_s(s) + od_B(s) = od_D(s) \quad \text{and}$$
$$id_s(s) + id_B(s) = id_D(s)$$

The *index of out-liaison concentration of a point s* is the ratio $od_B(s)/od_D(s)$. This index gives the proportion of arcs originating at s that are out-liaisons and thus the proportion of points adjacent to s that lie in the environment of S. Clearly, it is greater than zero if and only if s is in the out-frontier of S. The *index of in-liaison concentration of s* is the ratio $id_B(s)/id_D(s)$ and has an analogous meaning. Referring back to the system shown in Figure 1, we see that the indices of out-liaison and in-liaison concentration for s_1 are both $1/3$. Thus, this element lies in both the in-frontier and out-frontier of S, and $1/3$ of its contacts are outside of S. On the other hand, these indices are both $0/2$ for s_2 and s_3 whose contacts lie entirely within S.

Stratification of a System and Its Environment

In the discussion thus far, a distinction has been made between two types of points within a system S: frontier points (those adjacent with a point of E) and inner points (all other points of S). We now develop a more general method by which the points of S can be assigned to strata according to their proximity to

E. To do so, we need two directionally dual definitions of point-set distance. Let U be a subset of the set of points of D. Then the *distance* $d(v, U)$ *from a point* v *to a set* U is the minimum distance from v to a point in U, and *the distance* $d(U, v)$ *from* U *to* v is the minimum distance from a point u in U to v.[3] From a point s of S, the distance $d(s, N_o)$ is thus the minimum distance from s to any point in the out-neighborhood of S, and the distance $d(N_i, s)$ is the minimum distance to s from any point in the in-neighborhood of S. And since these distances are clearly the shortest distances between s and any point of E, they may be thought of as the distances between s and the environment of S, explicitly, $d(s, N_o) = d(s, E)$ and $d(N_i, s) = d(E, s)$ for all points s of S.

A *geodesic from a point* v *to a set* U is a path of minimum length from v to any point u in U. Thus, if s is a point in S, then each geodesic from s to N_o is a shortest path from s to any point in its out-neighborhood. A geodesic from a set of points to a single point is defined similarly. Thus each geodesic from N_i to s is a shortest path from any point in N_i to s. Clearly, $d(s, N_o) = k$ if and only if D contains a geodesic from s to N_o of length k, and $d(N_i, s) = k$ if and only if D contains a geodesic from N_i to s of length k.

It will be recalled that the boundary $B(S, D)$ is the subgraph of D that is induced by the liaisons of S. Let us now consider two analogous subgraphs of D that are induced by the geodesics from s to N_o and from N_i to s, respectively. The *out-stratification subgraph* $Z_o(S, D)$ is the subgraph of D induced by the geodesics from s to N_o, for all points s in S, and the *in-stratification subgraph* $Z_i(S, D)$ is the subgraph of D induced by the geodesics from N_i to the points s in S. Thus the point set of $Z_o(S, D)$ consists of N_o and the set S_o of all points of S that can reach N_o, and its arc set consists of just those contained in a geodesic from a point of S_o to the out-neighborhood N_o. Similarly, the point set of $Z_i(S, D)$ consists of N_i and the set S_i of all points of S that are reachable from N_i, and its arc set consists of just those contained in a geodesic from N_i to a point s of S_i. It should be noted that the boundary $B(S, D)$ is a subgraph of the union of $Z_o(S, D)$ and $Z_i(S, D)$. Figure 4 shows the stratification subgraphs for the system $S = \{s_1, s_2, s_3, s_4, s_5\}$.

We now define the *k'th out-stratum of* S, denoted ${}_kS_o$, as the subset of the points of S that can reach a point of N_o for which $d(s, N_o) = k$, and the *k'th in-stratum of* S, ${}_kS_i$, as the subset of the points s of S that are reachable from a point of N_i for which $d(N, s) = k$. Thus, the k'th out-stratum consists of those points of S that can reach a point of N_o by a geodesic of length k but not by any shorter path. And the k'th in-stratum consists of those points of S that can be reached from a point of N_i by a geodesic of length k but not by any shorter

[3]The definition of point-set distance gives precise meaning to the concept of distance from a point to the outside, which Lewin (1941) employed to characterize the stratification of "natural wholes."

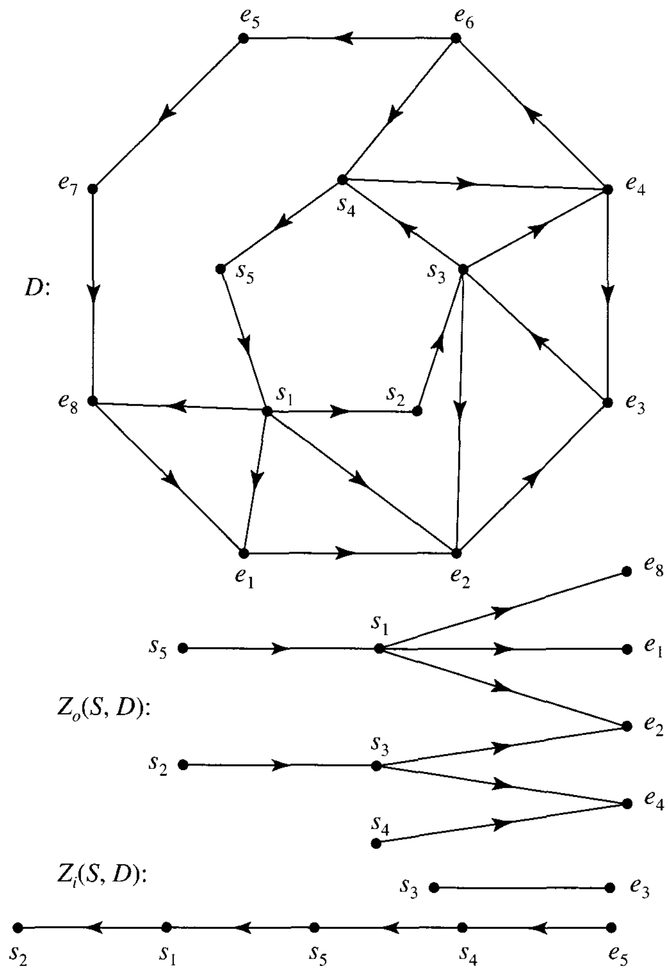


Figure 4. The stratification subgraphs $Z_o(S, D)$ and $Z_i(S, D)$ of a system S.

path. Clearly, the first out-stratum ${}_1S_o$ is the out-frontier of S, and ${}_1S_i$ is the in-frontier of S. It should be noted that no point of S lies in more than one out-stratum (or in-stratum), but there may be points that are not in any out-stratum because they cannot reach a point of N_o (or not in any in-stratum because they cannot be reached from a point of N_i). If, however, S is strong, then both the out-strata and the in-strata do partition the points of S.

The *degree of out-stratification of S* is the number of its out-strata, and the *degree of in-stratification of S* is the number of its in-strata. Clearly, the degree of out-stratification and the degree of in-stratification for a given system S need not be the same. If S contains m points, both of these indices may vary between 0 and m. The degree of out-stratification of S is 0 if and only if the out-frontier F_o is empty. It is 1 if and only if every point of S is in F_o. And it is m if and only if the following conditions are satisfied: (1) F_o consists of a single point s; (2) there exists a path W to s containing all points of S and no other ones and (3) every path from a point of S to s is a subpath of W. These three conditions together ensure that each point of S constitutes an out-stratum, and conversely. Corresponding statements about the degree of in-stratification can readily be formulated.

Referring again to Figure 4, we see that S has two out-strata, ${}_1S_o = F_o = \{s_1, s_3, s_4\}$ and ${}_2S_o = \{s_2, s_5\}$. Thus, its degree of out-stratification is 2. And since every point of S is contained in $Z_o(S, D)$, they can all reach a point of E in two steps or less along the geodesics contained in $Z_o(S, D)$. On the other hand, the degree of in-stratification of S is 4, and its in-strata are ${}_1S_i = F_i = \{s_3, s_4\}$, ${}_2S_i = \{S_3\}$, ${}_3S_i = \{s_1\}$ and ${}_4S_i = \{s_2\}$. And since every point of S is contained in $Z_i(S, D)$, they can all be reached by a point of E, but here 4 steps are required.

Let us assume that this digraph represents an interpersonal communication structure in which one unit of time is required for a message to traverse each communication link. Then each member in the k'th out-stratum of subgroup S can send a message to some nonmember in no more than k units of time, and everyone can do so in at most 2 units. By similar reasoning, it requires at least 4 units of time for a message to reach every member of the subgroup from the outside. If we make the further assumption that a message undergoes a unit of distortion each time it traverses a communication link, then any message originating with a member in the k'th out-stratum will have undergone at least k units of distortion when it first reaches a nonmember. And any message originating with a nonmember will have undergone at least k units of distortion when it reaches a member in the k'th in-stratum. Other implications of the stratification of a system can be readily derived by assigning other meanings to the terms point, arc and message, or by postulating other consequences of the transmission of messages.

It is evident that the points in the environment of a system can be stratified in a similar fashion. The *k'th out-stratum of E* (relative to S), denoted ${}_kE_o$, is the subset of the set of points of E that are reachable from a point of F_o for which $d(F_o, e) = k$, and the *k'th in-stratum of E*, ${}_kE_i$, is the subset of the points of E that can reach a point of F_i for which $d(e, F_i) = k$. The *degree of out-stratification* and the *degree of in-stratification of E* are the number of its out-strata and in-strata, respectively. For the digraph shown in Figure 4, the degree of out-stratification of E is 4 and its degree of in-stratification is 6.

Boundaries in Different Kinds of Digraphs

The concepts discussed thus far are applicable to digraphs in general. If, however, it is known that D is a digraph of some particular type, certain implications may be drawn concerning the boundary properties of systems contained within D. We now consider some of these implications for three common types of structures: symmetric digraphs, transitive digraphs and signed graphs.

For this purpose, we need to introduce some additional basic concepts of graph theory. A *semipath joining* v_1 *with* v_n is a collection of distinct points, v_1, $v_2, \ldots, v_n$, together with $n - 1$ arcs, one from each pair v_1v_2 or v_2v_1, v_2v_3 or $v_3v_2, \ldots, v_{n-1}v_n$ or v_nv_{n-1}. A digraph D is *weak* if every two points of D are joined by a semipath, and it is *strong* if every two points are mutually reachable. For any digraph D, a subgraph of D is *maximal* with respect to some property if D has no larger subgraph containing it which has this property. A *weak component* of D is a maximal weak subgraph. A *strong component* is a maximal strong subgraph. And a *clique* is a maximal complete symmetric subgraph.

Symmetric Digraphs

An arc uv of a digraph D is said to be *symmetric* if the arc vu is also in D; otherwise, it is said to be *asymmetric*. A *symmetric digraph* is one containing only symmetric arcs. Thus, for example, a communication structure is symmetric if all of its links are two-way, and a sociometric structure is symmetric if all choices are mutual. It is evident that if two points, u and v, are joined by a semipath in a symmetric digraph, they are also joined by two paths of the same length, one from u to v and one from v to u. Thus, every weak component of a symmetric digraph is strong, and the distance $d(u, v)$ equals $d(v, u)$ for any two points of D.

Let us now consider a system S within a symmetric digraph D. Clearly, every out-liaison s_ie_j of S has a corresponding in-liaison e_js_i, and conversely. Thus every liaison of S is a symmetric arc. It follows immediately that every point in the frontier F of S is in both its out-frontier F_o and in-frontier F_i, and every point in N is in both N_o and N_i. And since the distance between every pair of points is symmetric, the k'th out-stratum ${}_kS_o$ and the k'th in-stratum ${}_kS_i$ contain exactly the same set of points, so that the degrees of out-stratification and in-stratification are the same. Similar observations obviously hold for the stratification of the environment of S.

If D is a symmetric digraph and S is a strong component of D, then it can easily be shown that the boundary $B(S, D)$ must be empty. Thus, if a system S is a strong component of a symmetric structure, then messages can go, directly or indirectly, from each element to every other element of S but no messages can go in either direction between an element of S and an element of E.

TRANSITIVE DIGRAPHS

A digraph D is *transitive* if it contains an arc uw whenever arcs uv and vw are in D, for any distinct points u, v and w. Thus, for example, an interpersonal power structure is transitive if, for any three people, person u has power over w whenever u has power over v and v has power over w. And a sociometric structure is transitive if all members follow the principle: "A friend of my friend is my friend." A useful characterization of transitive digraphs is given in the following theorem (Harary, Norman and Cartwright, 1965: 120):

Theorem I. *A digraph is transitive if and only if it has the following properties:*
(1) Every strong component S is maximal complete symmetric (that is, a clique).
(2) There is an arc from every point in clique S_i to every point is S_j whenever there is a path from any point in S_i to any point in S_j.

Some of the boundary properties of cliques in transitive digraphs may now be specified:

Corollary 1a. *Let D be a transitive digraph. Then*
(1) Every symmetric arc of D is an internal arc of some clique S, and every asymmetric arc is a liaison of some clique.
(2) The out-frontier of any clique S is either empty or contains every point of S, and the same holds for its in-frontier.
(3) The out-neighborhood of any clique S consists of all points in its environment that are reachable from any point of S, and the in-neighborhood consists of all points in E that can reach any point of S.

Corollary 1b. *Let S_i and S_j be any two cliques of a transitive digraph. Then, in the subgraph induced by these two cliques the boundary of S_i satisfies exactly one of the following conditions: (1) it is empty; (2) it has an outward orientation and contains an out-liaison from every point of S_i to every point of S_j; (3) it has an inward orientation and contains an in-liaison from every point of S_j to every point of S_i.*

The *condensation* D^* of a digraph D is the digraph whose points correspond to the strong components $S_1, S_2, \ldots, S_n$ of D and whose arcs are determined as follows: there is an arc from point S_i to S_j in D^* if and only if there is at least one arc from a point of S_i to one of S_j in D. Clearly, every condensed digraph D^* is acyclic (and hence asymmetric), and each point S_i of D^* can therefore be assigned a level b_i such that $b_i < b_j$ for each arc S_iS_j in D^* (Harary, Norman and Cartwright, 1965:268). Now, if D is transitive, then each point S_i of D^* corresponds to a clique S_i of D, and the level b_i of S_i in D^* gives the level of its associated clique in D. Thus, for any clique S_i of a transitive digraph, each of its internal arcs joins two points at the same level, each of its out-liaisons goes to a point at a higher level and each of its in-liaisons comes from a point at a lower level. And since the condensation of a transitive digraph is transitive and asymmetric, we see that a transitive structure may be characterized as a partial order (or "hierarchy") of cliques.

It has been proposed by Davis (1967), Davis and Leinhardt (1970) and Holland and Leinhardt (1970; 1971) that sociometric choices display a tendency toward transitivity. If this tendency is fully realized by a given group of people, the resulting sociometric structure has the properties discussed above. The group thus consists of subgroups in which all members mutually choose each other. These subgroups can be assigned levels (perhaps indicative of their sociometric "status") such that all choices between members of different subgroups are unreciprocated and go from a lower to a higher level. And for every pair of subgroups S_i and S_j, either every member of S_i chooses every member of S_j and no member of S_j chooses any member of S_i, or there are no choices whatsoever between members of S_i and S_j.

The properties of transitive digraphs make it clear that as the number of people in a sociometric structure increases, the attainment of transitivity may require an increasingly large number of choices on the part of some people. If there are m members of a clique S, then each member of S must choose the $m - 1$ other members of S. And if there are r people in the environment of S who can be reached from any member of S, then every member of S must also choose r members of E. Thus the quantity $r + m - 1$ gives the number of choices required of each member of a clique S in a transitive structure. And it is clear that this number may be quite large if S is large or is located at a lower level of the structure. We might expect, therefore, that even if there is a tendency toward transitivity in such structures, they may not always contain all of the arcs required by transitivity due to some empirical constraint on the number of choices made by a given person.

This observation suggests that it would be useful to have a characterization of digraphs that approximate, but do not necessarily fully satisfy, the property of transitivity. We shall say that digraph is *quasi-transitive* if it has at least one arc from a point of a strong component S_i to one of S_j whenever there is a path from a point of S_i to one of S_j. Clearly, every transitive digraph is quasi-transitive, but not conversely. It can be easily shown that the condensation D^* of

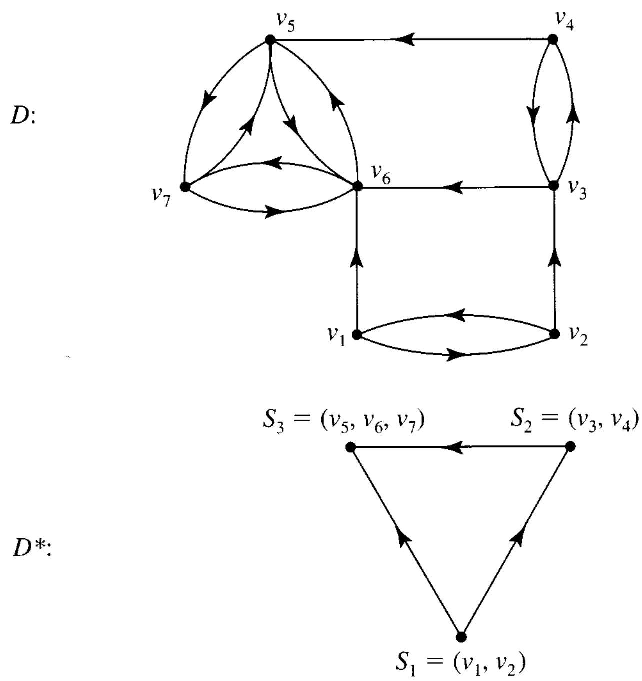


Figure 5. A quasi-transitive digraph D and its transitive condensation D^*.

a quasi-transitive digraph D is transitive and, of course, asymmetric. Thus, quasi-transitive and transitive digraphs are alike in that both may be characterized as a partial order of strong components and, in both, each strong component S_i can be assigned to a level b_i such that no arc goes from a higher level to a lower one. Figure 5 shows a quasi-transitive digraph D and its transitive condensation D^*.

Transitive and quasi-transitive digraphs differ essentially in their requirements for the number of internal arcs within each strong component and the number of liaisons between strong components. In a transitive digraph, the strong components are complete symmetric, but in a quasi-transitive digraph they may be neither complete nor symmetric. Thus if a strong component S_i of D contains m_i points, the number of internal arcs of S_i is $m_i\,(m_i - 1)$ when D is transitive, but this number may be as small as m_i when D is quasi-transitive. And if S_i and S_j are (nontrivial) strong components of D, the number of liaisons in a nonempty boundary between them is $m_i m_j$ when D is transitive, but this number may be as small as 1 when D is quasi-transitive.

Signed Digraphs

A *signed digraph* is one in which every arc has either a positive or a negative sign. Such digraphs may be used to represent any structure containing relationships that can be considered as intrinsically positive or negative, as, for example, interpersonal liking or disliking. Most research employing signed digraphs has been designed to test a hypothesis advanced by Heider (1946), Newcomb (1953; 1963), Davis (1967) and others that such structures display a tendency toward balance or clusterability. A signed digraph is *clusterable* if its set of points can be partitioned into subsets, called clusters, such that all points joined by a positive arc are in the same subset and all points joined by a negative arc are in different subsets. A signed digraph is *balanced* if it is clusterable into two clusters (Cartwright and Harary, 1956; 1968). The hypothesized tendency toward clusterability (or balance) thus implies a tendency for the elements of a signed structure to form clusters which contain only positive relationships.

Let us now consider the boundary properties of the clusters of a clusterable signed digraph D. It is immediately evident that all positive arcs of D are internal arcs of some cluster and that all negative arcs are liaisons. Thus, the boundary of each cluster of D contains only negative liaisons. We shall say that such a boundary is *negative*. The next theorem and corollary summarize these observations.

Theorem 2. *A signed digraph D is clusterable if and only if its set of points can be partitioned into clusters such that the boundary of each cluster is either empty or negative.*

Corollary 2a. *In a clusterable signed digraph, every point that is incident with a negative arc lies in the frontier of some cluster, and the frontier of every cluster contains only such points.*

We conclude this section by considering the properties of sociometric structures that are not only clusterable but also satisfy the requirement that positive choices are transitive or quasi-transitive. Let D^+ be the spanning subgraph obtained by removing all negative arcs from a signed digraph D, and let us call its weak components the *positive weak components* of D. The next theorem characterizes any clusterable signed digraph whose positive arcs display the property of quasi-transitivity. Statement (2) of this theorem is a consequence of a criterion for clusterability given by Davis (1967) and Cartwright and Harary (1968).

Theorem 3. *Let D be a clusterable signed digraph whose subgraph D^+ is quasi-transitive. Then for each positive weak component C of D,*
(1) C is a partial order of strong components.

(2) No negative arc of D joins two points of C.
(3) The boundary of C is either empty or negative.

If a signed sociometric structure realizes both of these tendencies, toward clusterability and quasi-transitivity (or transitivity), then all negative choices will be liaisons of some positive weak component and every positive weak component will be a partial order of its strong components. It should be noted that the important components are different for quasi-transitivity and clusterability. For the former, the strong components are critical and their liaisons are asymmetric arcs while for the latter, the weak components are critical and their liaisons are negative arcs.

Convexity in Digraphs

In the context of plane geometry, a set of points in the Euclidean plane is said to be convex if for any two points in the set the straight-line segment joining them lies within the set. Pfaltz (1971) has generalized this concept to directed graphs by defining a subset S of the points of a digraph D as *convex* if for any two points in S all walks in D from one to other have all their points in S.

It can readily be seen that the concept of convexity is intimately related to that of boundary, for if a walk joining two points of a system S contains a point not in S, it must contain both an out-liaison and an in-liaison of S. Thus, if the boundary of S is empty, outwardly oriented or inwardly oriented, the set of points in S must be convex since there can be no such walk in D. However, none of these sufficient conditions for convexity is necessary since S may be convex even though its boundary contains both out-liaisons and in-liaisons.

Let us assume that S is a convex subgroup within an interpersonal communication structure. Then it follows that if a message can go between two members of S it must stay entirely within the subgroup. And, if we assume that S is a convex department within the mobility structure of an organization, then no permissible career line between two positions of S will contain a position outside of the department. From these two examples, we see that convexity reflects a kind of segregation of a system within a structure and that measures of the degree of convexity will reflect corresponding degrees of segregation. It should be noted, however, that a convex system may influence its environment or be influenced by it.

Given a specified set S of the points of a digraph D, we say that a walk is internal to S, or an *internal walk*, if all of its points are in S. An *excursion from* S is a walk whose first arc is an out-liaison of S, whose last arc is an in-liaison of S, and which contains no other liaisons. Thus, every excursion from S starts at a point in the out-frontier of S, passes through a point in its out-neighborhood and one in its in-neighborhood and ends at a point in its in-frontier. The next theorem characterizes the convex sets in any digraph D.

Theorem 4. *The following statements are equivalent for any set S of points of a digraph D:*
(1) S is convex.
(2) D contains no walk from a point in the out-neighborhood of S to a point in its in-neighborhood. Thus, no point in N_o can reach a point in N_i.
(3) D contains no excursions from S.

Corollary 4a. *The points in each strong component of D form a convex set.*

As an illustration of this theorem, let us consider the system $S = \{s_1, s_2, s_3\}$ contained in the digraph D of Figure 6. Clearly, S is a strong component of D and, in keeping with Corollary 4a, it is convex since all walks between any two of its points lie entirely within S. Statement (2) is satisfied since $N_o = \{e_1\}$, $N_i = \{e_4\}$ and there is no walk from e_1 to e_4. And Statement (3) is satisfied since D contains no excursion from S.

The digraph shown in Figure 7 is the same as the one in Figure 6 except that the arc e_1e_4 is substituted for e_4e_1. Now, S is not convex since e_1 can reach e_4, and there is an excursion from s_2 to s_3 via e_1 and e_4.

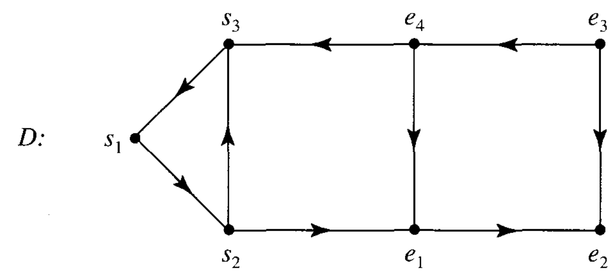


Figure 6. A convex system S.

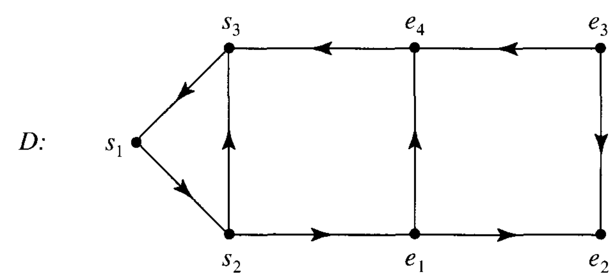


Figure 7. A system S that is not convex.

If a subset S of points of a digraph D is not convex, then it must be contained in a convex set since the set V of all points of D is obviously convex. The *convex hull* of S, denoted $H(S)$, is the smallest convex set of points of D containing S. Clearly, S is convex if and only if $H(S) = S$. The next theorem, whose proof is omitted, establishes that every set of points of a digraph has exactly one convex hull.

Theorem 5. *For any set S of points of D, the convex hull of S is unique.*

Referring again to the digraph of Figure 7, we see that the convex hull of S is $H(S) = \{s_1, s_2, s_3, e_1, e_4\}$. Points e_1 and e_4, and no others, must be added to S since these points are contained in the unique excursion from S.

If a point u is not the only point in its strong component, the set $S = \{u\}$ is not convex, for then there is at least one other point v mutually reachable from u which thus lies on a walk from u to itself. This observation, together with Corollary 4a, shows that the strong component containing u is precisely the convex hull of $\{u\}$. The next theorem follows directly.

Theorem 6. *The convex hull of a set S of points of D is the union of the strong components containing each of its points.*

This theorem now provides a means of identifying all of the convex sets in a digraph D. It will be recalled that the strong components C_i of D are the points of the condensation D^*.

Corollary 6a. *If* $\{C_1, C_2, \ldots, C_n\}$ *is a convex set in* D^**, then the points in* $C_1 U C_2 U \ldots U C_n$ *form a convex set in D, and conversely.*

The *extension of S*, denoted $W(S)$, is the subgraph of D induced by all the excursions from S. Clearly, S is convex if and only if $W(S)$ is empty, and the union of the set of points in $W(S)$ with those in S constitutes the convex hull $H(S)$. Figure 8 shows a system $S = \{s_1, s_2, s_3\}$ and its extension $W(S)$. It can readily be seen that S is not convex and that its convex hull is $H(S) = \{s_1, s_2, s_3, e_1, e_2, e_5\}$.

The extension of a system S that is not convex may be thought of as the subgraph of D "responsible" for this lack of convexity since $W(S)$ contains all excursions from S. This observation suggests four different indices of the degree to which a system S deviates from convexity. The first two are provided by the size of $W(S)$, as measured by the number of its points or arcs. For the system shown in Figure 8, these indices are both 5. A third index, more structural in nature, is given by the number of excursions contained in $W(S)$. There is, however, a problem in counting these excursions since any digraph contain-

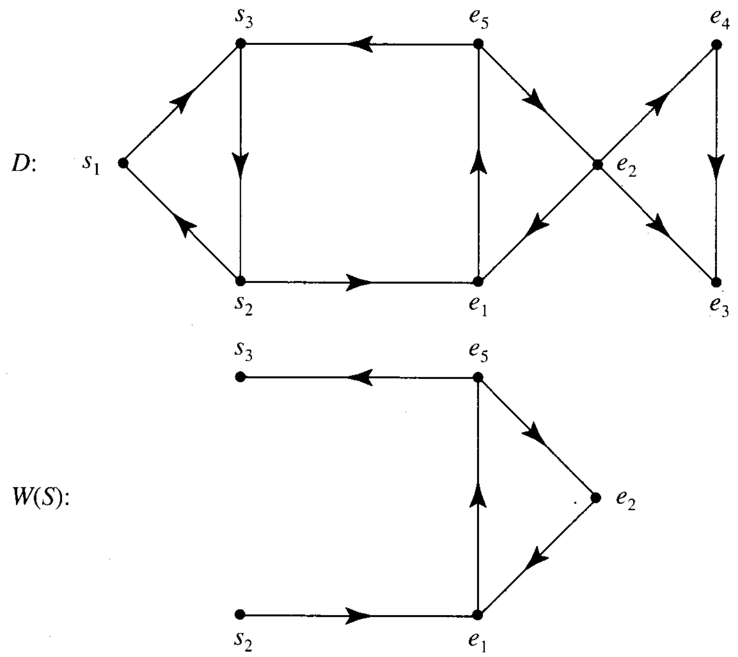


Figure 8. A system S and its extension $W(S)$.

ing a cycle has an infinite number of walks (of finite but unlimited length). Thus, if $W(S)$ contains a cycle, a meaningful index can be obtained only by counting walks of restricted length. For the example given in Figure 8, it would seem natural to consider excursions of length 6 or less, and we see that there are two such excursions, namely, one of length 3 and one of length 6. The fourth index is provided by the *hull-ratio* of S, which is defined as the number of points in S divided by the number in $H(S)$. Since the number of points in $H(S)$ is clearly the number in S plus the number in $W(S)$ not contained in S, the hull-ratio of S is 1 if and only if S itself is convex and decreases as more and more points are required to form a convex set from S. For the system shown in Figure 8, we see that the hull-ratio of S is $3/6$.

The excursions of any system S may, of course, penetrate the environment of S in varying degrees. A measure of the depth of environmental penetration of these excursions can be constructed in terms of the out-strata ${}_nE_o$ of the environment of S, described above. More specifically, we define the *out-reach* of S as the maximum n such that $W(S)$ contains a point in ${}_nE_o$, the n'th out-strata of S. Referring again to Figure 8, we see that the out-strata of the environment of S are ${}_1E_o = \{e_1\}$, ${}_2E_o = \{e_5\}$, ${}_3E_o = \{e_2\}$ and ${}_4E_o = \{e_3, e_4\}$. And since $W(S)$ contains e_2 but not e_3 or e_4, the out-reach of S is 3.

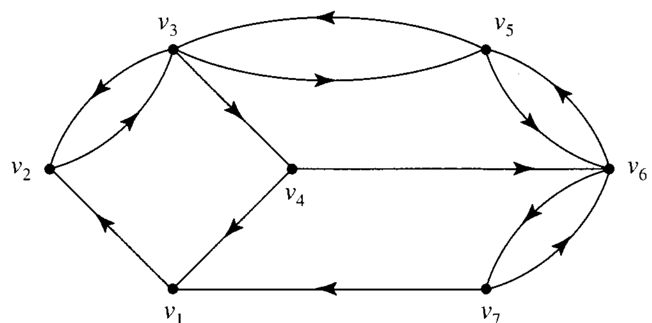
Matrix Operations

In empirical research on boundaries and convexity, the digraphs considered will usually be sufficiently complex to make it difficult to ascertain their properties by visual inspection. We now indicate briefly how the adjacency matrix of a digraph can be used to facilitate the investigation of these properties. Given a digraph D, its *adjacency matrix* $A(D)$ is a square matrix with one row and one column for each point of D, in which $a_{ij} = 1$ if arc v_iv_j is in D and $a_{ij} = 0$ if v_iv_j is not in D. Figure 9 shows a labeled digraph D and its adjacency matrix $A(D)$, where $S = \{v_1, v_2, v_3, v_4\}$ is the set of points of system S and $E = \{v_5, v_6, v_7\}$ is the set of points in its environment. We assume for present purposes that the points of D are labeled so that those in S are $v_1, v_2, \ldots, v_m$ and those in E are $v_{m+1}, v_{m+2}, \ldots, v_p$.

It will be noted that the matrix $A(D)$ in Figure 9 is partitioned into four submatrices which can be denoted as follows:

$$A(D) \quad \left[\begin{array}{c|c} A(S) & A(L_o) \\ \hline A(L_i) & A(E) \end{array}\right]$$

D:



$$A(D) = \begin{array}{c} \\ S \\ \\ \\ E \\ \end{array} \overset{\qquad S \qquad\qquad\quad E}{\left[\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]}$$

Figure 9. The adjacency matrix of a digraph containing a system S and its environment E.

Let us consider the entries in each of these submatrices separately. The unit entries in $A(S)$ clearly correspond to the internal arcs of S, since $A(S)$ is the adjacency matrix of S. The unit entries in $A(L_o)$ correspond to the out-liaisons of S, since their first points are in S and their second points are in E. The unit entries in $A(L_i)$ correspond to the in-liaisons of S, since their first points are in E and their second points are in S. And the unit entries in $A(E)$ correspond to arcs lying entirely in E.

It can readily be ascertained that in this example $L_o = \{v_3v_5, v_4v_6\}$, $L_i = \{v_5v_3, v_7v_1\}$, $F_o = \{v_3, v_4\}$, $F_i = \{v_1, v_3\}$, $N_o = \{v_5, v_6\}$ and $N_i = \{v_5, v_7\}$. From this information, the boundary $B(S, D)$ can be immediately identified.

In general, the boundary of a system S within a digraph D is empty if and only if the submatrices $A(L_o)$ and $A(L_i)$ contain only zeros. It is outwardly oriented if and only if $A(L_o)$ has at least one unit entry and $A(L_i)$ has none. And it is inwardly oriented if and only if $A(L_i)$ contains a unit entry while $A(L_o)$ does not.

We now show how the adjacency matrix $A(D)$ can be employed to provide information about the convexity properties of any system S within D. For this purpose, we make use of the fact that each entry $a_{ij}^{(k)}$ in the k'th power of $A(D)$ gives the number of walks of length k from v_i to v_j in D. Starting with a given adjacency matrix $A(D)$ we change each unit entry in the submatrix $A(S)$ to zero. This has the effect of forming the adjacency matrix $C = A(D^0)$ of the subgraph D^0 obtained by deleting from D all internal arcs of S. Clearly, every walk from a point v_i to v_j of S in D^0 is an excursion from S in D, since none of these walks contains an internal arc of S. The next theorem follows immediately.

Theorem 7. *Each entry* $c_{ij}^{(k)}$ *in the submatrix* $C^k(S)$ *of the k'th power of C gives the number of excursions of length k from a point* v_i *to a point* v_j *in S.*

This result, together with Theorem 4, leads directly to the following corollary.

Corollary 7a. *The set S of points is convex in a digraph D if and only if in all powers* C^k *of the matrix C, the sub-matrix* $C^k(S)$ *has only zero entries.*

Summary and Conclusions

The conceptualization presented in this paper is intended to apply to any empirical structure consisting of a set of elements and a set of relationships, where some of the elements can be conceived as belonging to a *system* and the remainder to its *environment*. For any such structure, there is an associated digraph D whose points and arcs represent the elements and relationships of the structure, and whose set of points is partitioned into two subsets, *S* (those corresponding to the elements of the system) and *E* (those in its environment). Given this partitioning, four types of arcs (relationships) are distinguished according to whether they join (a) two points of *S* (internal arcs), (b) two points of *E* (external arcs), (c) a point of *S* to one of *E* (out-liaisons of *S*) or (d) a point of *E* to one of *S* (in-liaisons of *S*). The boundary of *S* is then defined as the subgraph of *D* induced by the liaisons of *S*. The boundary thus consists of those elements of *S* that are in its frontier, those elements of *E* in its neighborhood and those relationships of *D* that join elements of *S* with ones in *E*. Finally, these concepts, together with others of graph theory, are used to define a number of higher-order concepts and related indexes, such as the orientation of a boundary, the degree of stratification of a system and its environment and the convexity of a system.

The strict correspondence between the elements and relationships of a given structure and the points and arcs of its associated digraph *D* assures that the formal properties of *D* are also properties of this empirical structure. It should be noted that these properties, in themselves, refer only to the structural aspects of system-environment relationships and not to the processes occurring in such structures. We believe, however, that the approach adopted here is entirely compatible with more dynamically oriented work of systems theorists and could be extended to incorporate their findings. Thus, for example, Emery and Trist (1965/Vol. III), writing in the context of organizational theory, argue that a basic distinction should be made among four fundamentally different types of processes: (a) those occurring between elements within an organization (system); (b) those between elements in its environment; (c) those directed from an element of the organization to one in its environment and (d) those directed from an element of the environment to one of the system. The isomorphism between this classification of processes and our classification of arcs suggests that distinctive processes are associated with each type of arc of an empirical structure. And a specification of the exact nature of these processes should permit the formulation of a number of detailed hypotheses as to how the structural properties of a system and its environment are related to the processes occurring in such a structure.

As a first step toward clarifying the relations between these structural properties and their associated processes, we have proposed that such processes be conceived as involving the transmission of "messages" through the arcs (links) of a structure. And we have suggested some ways in which messages may be differentially affected when transmitted through the different types of arcs. Thus, for example, we have proposed that the content or meaning of a message may be altered to a larger degree or in a systematically different way when it traverses a liaison as compared to a link that is internal or external to a system. And two somewhat different effects are suggested by Miller (1971) who proposes that (a) more work is required to transmit a message over the boundary of a system than to transmit it the same distance immediately inside or outside the boundary and (b) the amount of information transmitted between elements within a system is significantly larger than the amount transmitted across the boundary. A more fully developed analysis of the ways in which messages are modified as they are transmitted through the arcs of a structure awaits further theoretical and empirical research.

In conclusion, we note the possibility of extending our formalization by considering processes that occur "within" the elements of a structure as opposed to its links. A beginning in this direction might make use of the work, cited above, on the distinctive behavior of "boundary persons" (those located in the frontier of *S*) and the suggestive ideas of Lewin concerning the functions performed by "gatekeepers." Once the exact nature of the processes occurring at the elements and links of a structure have been identified, it should then be possible to employ the results of our conceptualization to derive a variety of empirically testable hypotheses about the dynamics of system-environment relationships.

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